A LIMITING PROCESS IN THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM

(O PREDEL'NOM PEREKHODE V RESHENII ODNOI ZADACHI OPTIMAL'NOGO REGULIROVANIIA)

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F. M. KIRILLOVA
(Sverdlovsk)

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An optimal problem [1-6] is considered for a linear system of differential equations in which the integrals of the pth power ($p > 1$) of the modulus of the control function are bounded.

1. Let the control system be described by the equation

$$\frac{dx}{dt} = A(t)x + b(t)u(t) \tag{1.1}$$

where $x = \{x_1(t), \ldots, x_n(t)\}$ (the representative vector in the phase space), the elements $a_{ik}(t)$ ($i = 1, \ldots, n$, $k = 1, \ldots, n$) of the matrix $A(t)$, and the components $b_i(t)$ ($i = 1, \ldots, n$) of the vector $b(t)$ are continuous functions of time t.

We shall assume that the control function $u(t)$ satisfies the condition

$$\int_{t_0}^{t} |u(\tau)|^p d\tau \leqslant 1 \qquad (p > 1) \tag{1.2}$$

The optimal problem [1-6] consists of the following: among all control functions satisfying the condition (1.2), it is required to find such a $u(r, p)$ that a point moving along the trajectory of Equation (1.1) will move from the initial point $x(t_0) = x_0$ to the origin of the coordinate system in the shortest time $T(p)$.

This time $T(p)$ is called the optimal time of the transfer process. The corresponding control $u(r, p)$ is called the optimal control function.

The condition (1.2), with $p = 2$, corresponds to a limitation on the mean power of the controlling reaction $u(r)$. The investigation of the given problem under the restriction (1.2) with arbitrary $p > 1$, is of importance in going over to the problem with the restriction that

$$|u(\tau)| \leqslant 1 \qquad (t_0 \leqslant \tau \leqslant t) \tag{1.3}$$

Below, in Section 2, it is proved that the optimal control $u(\tau, p)$ for the case of restriction (1.2), which is continuous, converges in measure as $p \to \infty$ to the optimal control $u(\tau)$ of the same problem under condition (1.3). It is also shown that the function $T(p)$, with $p \to \infty$, has for a bound the time of the optimal process under the condition (1.3).

This circumstance guarantees the possibility of obtaining an approximate solution of the optimal problem under condition (1.3) by reducing it to the same problem under the restriction of condition (1.2). This latter problem can be solved by the usual methods of the calculus of variations as is well known.

Let us consider the space $L_q(t_0, t)$ of the functions $\phi(\tau)$ summable with the exponent q:

$$\int_{t_0}^{t} |\varphi(\tau)|^q d\tau < +\infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

By the symbol $\Lambda(t, q; \phi)$ we denote the norm of the function $\phi(\tau)$. This norm is defined in the following way:

$$\Lambda(t, q; \varphi) = \left(\int_{t_0}^{t} |\varphi(\tau)|^q d\tau\right)^{\frac{1}{q}}$$

As is shown in the work [7], the general form of linear functionals f defined on the space $L_q(t_0, t)$ is given by

$$f(\varphi) = \int_{t_0}^{t} \varphi(\tau)\eta(\tau)\,d\tau$$

Here, the function $\eta(t)$ satisfies the inequality

$$\Lambda(t, p, \eta) < +\infty$$

and the norm of the functional $\Lambda(t, p, f)$ of the functional f is defined by the equation

$$\Lambda(t, p, f) = \Lambda(t, p, \eta)$$

The solution of Equation (1.1) has the form [8]

$$x(t) = F(t)\,x_0 + \int_{t_0}^{t} F(t)\,F^{-1}(\tau)\,b(\tau)\,u(\tau)\,d\tau$$

where $F(t)$ is the fundamental matrix of Equation (1.1) with $B(t) = 0$. Therefore, if the point $x(t)$ arrives at the origin of the coordinate system at the time t, we have the following equation for the determination of $u(\tau)$:

$$-x_0 = \int_{t_0}^{t} F^{-1}(\tau)\, b(\tau)\, u(\tau)\, d\tau \tag{1.4}$$

In other words, the optimal problem has been reduced to the problem considered in the work [9]: among linear functionals f, defined on the space $L_q(t_0, t)$ it is required to find a function which will satisfy condition (1.2) and will be a solution of Equation (1.4).

We impose the following condition (A) on the equation (1.1). We assume that the function $\gamma(\tau)$ defined by the relation

$$\gamma(\tau) = (lF^{-1}(\tau)\, b(\tau)) \tag{1.5}$$

vanishes only at isolated points τ.

The symbol $(lF^{-1}(\tau)b(\tau))$ denotes the scalar product of the vector $l(l_i = \text{const},\ l_1^2 + \ldots + l_n^2 \neq 0)$ by the vector $F^{-1}(\tau)b(\tau)$.

It is shown in the work [9] that Equation (1.4) has a solution if

$$\Lambda(t, p, f) \geqslant \lambda(t, q) \qquad \left(\frac{1}{\lambda(t, q)} = \min \Lambda(t, q, \gamma) \ \text{ if } \ (lx_0) = -1\right)$$

We note that if condition (A) is satisfied then the

$$\min \Lambda(t,\ q,\ \gamma) \ \text{ with } \ (lx_0) = -1$$

is always attained [9], i.e. there exists a vector $l^{(q)}$ such that

$$\min \Lambda(t,\ q,\ \gamma) = \Lambda(t, q, \gamma^{(q)}) \tag{1.6}$$

We shall call a function $\gamma^{(q)}(\tau)$ which satisfies the condition (1.6) a minimizing element of the space $L_q(t_0, t)$ under the condition $(lx_0) = -1$.

Under fixed initial values x_0, the function $\lambda(t, q)$ is continuous and strictly monotone in t. This fact was proved in the work [6] when $q = 1$. We shall not give a proof of this property of the function $\lambda(t, q)$ when $q > 1$, for it would be a repetition (with slight changes) of the proof of the continuity and monotone nature of the function $\lambda(t, q)$ in t with $q = 1$.

Hence, if there exists at least one value t for which the inequality $\lambda(t, q) < 1$ is satisfied then the optimal problem has a unique solution. (One must take into consideration that the function $\lambda(t, q)$ is a decreasing function of t [6]). The optimal control time is found, obviously, from the relation $\lambda(t, q) = 1$.

Let us assume that the inequality $\lambda(t, q) < 1$ is valid, and let us find the optimal control function $u(\tau, p)$. Suppose that the function $\gamma^{(q)}(\tau)$ is the minimizing element under the condition $(lx_0) = -1$.

As is shown in the work [9], every minimizing element $\gamma^{(q)}(\tau)$ under condition $(lx_0) = -1$ is an extremal element for the functional f whose norm is $\lambda(t, q)$. Hence, if $\gamma^{(q)}(\tau)$ is a minimizing element under the condition $(lx_0) = -1$, and if $\Lambda(t, p, f) = \lambda(t, q)$, then we have

$$-(l^{(q)}x_0) = 1 = \int_{t_0}^{t} \gamma^{(q)}(\tau)\, u(\tau)\, d\tau = \lambda(t, q)\, \Lambda(t, q, \gamma^{(q)}) \tag{1.7}$$

We write down Holder's inequality for the functions $\gamma^{(q)}(\tau)$ and $u(\tau)$:

$$\int_{t_0}^{t} |\gamma^{(q)}(\tau)\, u(\tau)|\, d\tau \leqslant \Lambda(t, q, \gamma^{(q)})\, \Lambda(t, p, u) \tag{1.8}$$

From (1.7) and (1.8) it follows that the control vector $u(\tau)$ has the following property:

$$\operatorname{sign} u(\tau) = \operatorname{sign} \gamma^{(q)}(\tau)$$

Strictly speaking, this relation holds almost everywhere on the interval $t_0 \leqslant \tau \leqslant t$.

The equality sign can hold in (1.8) if, and only if, $|u(\tau)|^p = C|\gamma^{(q)}(\tau)|^q$. Hence, we have the following relation for the determination of the constant C by means of (1.8):

$$\int_{t_0}^{t} C^{-p} |\gamma^{(q)}(\tau)|^q d\tau = 1 \qquad \text{or} \qquad c = \Lambda^{-pq}(t, q, \gamma^{(q)})$$

The control functions $u(\tau)$ can, therefore, be found by means of the formula

$$u(\tau) = \lambda^q(t, q)\, |\gamma^{(q)}(\tau)|^{\frac{q}{p}} \operatorname{sign} \gamma^{(q)}(\tau) \tag{1.9}$$

The optimal control $u(t, p)$ can be found, as has already been pointed out, by setting $\lambda(t, q) = 1$ in (1.9). Hence, it will have the form

$$u(t, p) = |(l^{(q)}F^{-1}(t)\, b(t))|^{\frac{q}{p}} \operatorname{sign}(l^{(q)}F^{-1}(t)\, b(t)). \tag{1.10}$$

2. Let us consider the behavior of the function $\lambda(t, q)$ as $q \to 1$. This has to be done for the investigation of the optimal control $u(t, p)$

and the optimal time $T(p)$ as $p \to \infty$.

Lemma 2.1. If x_0, t are fixed and $q \to 1$, then the function $\lambda(t, q)$ has a limit $\lambda(t)$, and

$$\frac{1}{\lambda(t)} = \min \Lambda(t, 1, \gamma) \qquad \text{if } (lx_0) = -1 \tag{2.1}$$

Proof. Let q_s be an arbitrary decreasing sequence, and let $\lim q_s = 1$ as $s \to \infty$.

If

$$\frac{1}{\lambda(t, q_s)} = \Lambda(t, q_s, \gamma^{(q_s)}) \quad \text{if } (l^{(qs)}x_0) = -1 \tag{2.2}$$

$$\frac{1}{\lambda(t)} = \Lambda(t, 1, \gamma^\circ) \qquad \text{if } (l^\circ x_0) = -1 \tag{2.3}$$

then

$$\Lambda(t, q_s, \gamma^{(q_s)}) \leqslant \Lambda(t, q_r, \gamma^\circ) \tag{2.4}$$

But if $t \leqslant \tau \leqslant t$ we have by the relation (1.5) that $|\gamma^\circ(\tau)| \leqslant M$, where M is a constant. Hence

$$\Lambda(t, q_s, \gamma^{(q_s)}) \leqslant M_1 \tag{2.5}$$

where M_1 is a constant (for fixed t).

The function $\gamma(\tau)$ satisfies condition (A). One can show in this case that the sequence $l^{(q_s)}$ is bounded uniformly in s.

Hence, there exists a subsequence $l^{(q_{sm})}$ of the sequence $l^{(q_k)}$ such that

$$\lim l^{(q_{sm})} = l^{(1)} \qquad \text{as } m \to \infty, \qquad (l^{(1)}x_0) = -1$$

Furthermore, it is obvious that for fixed m the function $|\gamma^{(q_{sm})}(\tau)|^{q_{sm}}$ is continuous in τ, and that as $m \to \infty$

$$|\gamma^{(q_{sm})}(\tau)|^{q_{sm}} \to |\gamma^{(1)}(\tau)|$$

uniformly in τ. Therefore, the limiting process may be applied to the inequality

$$\Lambda(t, q_{sm}, \gamma^{(q_{sm})}) \leqslant \Lambda(t, q_{sm}, \gamma^\circ)$$

we obtain

$$\Lambda(t, 1, \gamma^{(1)}) \leqslant \Lambda(t, 1, \gamma^\circ) \tag{2.6}$$

On the other hand, we have

$$\Lambda(t, 1, \gamma^\circ) = \min \Lambda(t, 1, \gamma) \qquad \text{if } (lx_0) = -1$$

Hence, the following inequality holds:

$$\Lambda(t, 1, \gamma^{(1)}) \geqslant \Lambda(t, 1, \gamma^{\circ}) \tag{2.7}$$

From (2.6) and (2.8) it follows that

$$\Lambda(t, 1, \gamma^{(1)}) = \Lambda(t, 1, \gamma^{\circ})$$

This means, however, that the function $\gamma^{(1)}(\tau)$ is a minimizing element under condition $(lx_0) = -1$. The above argument is valid for an arbitrary sequence $\Lambda(t, q_s)$ and its subsequences. Hence, we have proved that $\lim \lambda(t, q) = \lambda(t)$ as $q \to 1$.

Let $u(t)$ be an optimal control and T be an optimal control time for the initial values x_0 with the restriction (1.3).

If the optimal problem is considered with the restriction (1.2), then, as above, the optimal control and optimal time will be denoted by $u(\tau, p)$ $T(p)$ respectively.

The following assertion is true:

Theorem 2.1. For each given $\epsilon > 0$ there exists an $N > 0$ such that for every $p > N$ the inequalities

$$|T(p) - T| < \varepsilon, \qquad \operatorname{mes} E(|u(\tau, p) - u(\tau)| \geqslant \sigma) < \varepsilon$$

are valid.*

Proof. Since for a fixed x_0 the function $\lambda(t)$ is continuous and monotone decreasing in t, it follows that for each $\epsilon > 0$ there exists a $\beta > 0$ such that the following inequalities hold:

$$\lambda(t - \varepsilon) > 1 + \beta, \qquad \lambda(t + \varepsilon) < 1 - \beta$$

Further, for fixed x_0, t, the function $\lambda(t, q)$ is continuous from the right in q.

Hence, for each given $\beta > 0$ there exists a $\delta_1 > 0$ such that for $q - 1 < \delta_1$ we have

$$|\lambda(t + \varepsilon, q) - \lambda(t + \varepsilon)| < \beta, \qquad |\lambda(t - \varepsilon, q) - \lambda(t - \varepsilon)| < \beta$$

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* The symbol mes $E(|u(\tau, p) - u(\tau)| > \sigma)$ means the measure of the set on which the inequality $|u(\tau, p) - u(\tau)| > \sigma$ holds, where σ is an arbitrary positive number.

From the given inequalities and from the continuity in t (for fixed q) of the function $\lambda(t, q)$ it now follows that if

$$q - 1 < \delta_1 \qquad \left(p > \frac{\delta_1}{\delta_1 - 1}\right)$$

then there exists a $T(p)$ for which

$$\lambda(T(p), q) = 1, \qquad |T(p) - T| < \varepsilon$$

Next, we show that

$$\operatorname{mes} E(|u(\tau, p) - u(\tau)| \geqslant \sigma) < \varepsilon \quad \text{if} \quad p > \frac{\delta}{\delta - 1}, \quad \delta > 0$$

In the works [2-6] it has been proved that the function $u(\tau)$ has the form

$$u(\tau) = \operatorname{sign}(l^\circ F^{-1}(\tau)\, b(\tau)) \qquad (t_0 \leqslant \tau \leqslant t_0 + T)$$

Let us consider the set of vectors $l^{(q)}$ from Formula (1.10).

As was pointed out in the proof of Lemma (2.1), the set of vectors $l^{(q)}$ is uniformly bounded in q. Furthermore, it is also uniformly bounded in $T(p)$ [10].

If $l^{(q_s)}$ is a convergent sequence and if

$$\lim l^{(q_s)} = l^{(1)} \quad \text{as} \quad s \to \infty$$

then one can show (see proof of Lemma 2.1) that the function $l^{(1)}F^{(-1)}(\tau)b(\tau)$ is a minimizing element under the condition

$$(lx_0) = -1$$

It is known [9] that

$$\operatorname{sign}(l^{(1)}F^{-1}(\tau)\, b(\tau)) = \operatorname{sign}(l^\circ F^{-1}(\tau)\, b(\tau))$$

Therefore, for every ϵ_1 neighborhood ($\epsilon_1 < \epsilon$) of the zeros of the function $(l^\circ F^{-1}(\tau)b(\tau))$ there exists a $\delta_2 > 0$ such that for $q_s - 1 < \delta_2$ the zeros of the function $(l^{(q_s)}F^{-1}(\tau)b(\tau))$ will lie in this ϵ_1 neighborhood of the zeros of the function $(l^\circ F^{-1}(\tau)b(\tau))$.

Outside such an ϵ_1-neighborhood the signs of the functions $(l^{(q_s)}F^{-1}(\tau)b(\tau))$ and $(l^\circ F^{-1}(\tau)b(\tau))$ coincide.

Since we have $q_s/p_s < \delta_3$, where δ_3 is a small positive number, it follows that

$$\operatorname{mes} E(|u(\tau, p) - u(\tau)| \geqslant \sigma) < \varepsilon_1 \quad \text{if} \quad q_s - 1 < \delta_3$$

Thus we obtain, finally

$$|T(p) - T| < \varepsilon, \ \mathrm{mes}\, E(|u(\tau, p) - u(\tau)| \geqslant \sigma) < \varepsilon$$

provided $p > N$ (N can be chosen to be the larger one of the numbers $\delta_1/(\delta_1 - 1)$ and $\delta_3/(\delta_3 - 1)$). The theorem has thus been proved.

It is possible to prove that all the arguments of Sections 1 and 2 are valid for the case of several control functions of Equation (1).

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